5.1 - Trigonometric Identities

Warmup

Establish the identity

1) $$\frac{\sin^2(-x) - \cos^2(-x)}{\sin(-x) - \cos(-x)} = \cos x - \sin x$$

2) $$\frac{1 + \tan x}{1 + \cot x} = \tan x$$

3) $$\frac{\sin x}{1 + \cos x} + \frac{1 + \cos x}{\sin x} = 2\csc x$$

5.2 - Sum and Difference Identities

First Cofunction Identity

Start with the cosine difference identity

$$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

Substitute $\alpha = \dfrac{\pi}{2}$ and $\beta = \theta$

$$\cos(\tfrac{\pi}{2} - \theta) = \cos\tfrac{\pi}{2}\cos\theta + \sin\tfrac{\pi}{2}\sin\theta$$

$$\cos(\tfrac{\pi}{2} - \theta) = 0 \cdot \cos\theta + 1 \cdot \sin\theta$$

$$\boxed{\cos(\tfrac{\pi}{2} - \theta) = \sin\theta}$$

5.2 - Sum and Difference Identities

Second Cofunction Identity

Again, start with the cosine difference identity

$$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

Substitute $\alpha = \dfrac{\pi}{2}$ and $\beta = \dfrac{\pi}{2} - \theta$

$$\cos[\tfrac{\pi}{2} - (\tfrac{\pi}{2} - \theta)] = \cos\tfrac{\pi}{2}\cos(\tfrac{\pi}{2} - \theta) + \sin\tfrac{\pi}{2}\sin(\tfrac{\pi}{2} - \theta)$$

$$\cos\theta = 0 \cdot \cos(\tfrac{\pi}{2} - \theta) + 1 \cdot \sin(\tfrac{\pi}{2} - \theta)$$

$$\boxed{\cos\theta = \sin(\tfrac{\pi}{2} - \theta)}$$

$$\boxed{\cos(\tfrac{\pi}{2} - \theta) = \sin\theta}$$

5.2 - Sum and Difference Identities

Deriving the Sine sum of angles

Start with the cofunction identity

$$\sin\theta = \cos(\tfrac{\pi}{2} - \theta)$$

Substitute $\theta = A + B$

$$\sin(A+B) = \cos(\tfrac{\pi}{2} - (A+B))$$

$$\sin(A+B) = \cos((\tfrac{\pi}{2} - A) - B)$$

$$\sin(A+B) = \cos(\tfrac{\pi}{2} - A)\cos B + \sin(\tfrac{\pi}{2} - A)\sin B$$

$$\boxed{\sin(A+B) = \sin A \cos B + \cos A \sin B}$$

5.2 - Sum and Difference Identities

Deriving the Sine difference of angles

Start with the Sine sum of angles

$$\sin(A + B) = \sin A \cos B + \cos A \sin B$$

Substitute B with $-B$

$$\sin(A + (-B)) = \sin A \cos(-B) + \cos A \sin(-B)$$

$$\sin(A - B) = \sin A \cos B - \cos A \sin B$$

5.2 - Sum and Difference Identities

$$\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$$
$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

Evaluate each of the following exactly:

$a.\ \sin 15°$ $\qquad \sin(45° - 30°) = \dfrac{\sqrt{6} - \sqrt{2}}{4}$

$b.\ \sin\left(\dfrac{7\pi}{12}\right)$ $\qquad \sin\left(\dfrac{4\pi}{12} + \dfrac{3\pi}{12}\right) = \dfrac{\sqrt{6} + \sqrt{2}}{4}$

$c.\ \sin 75°$ $\qquad \sin(120° - 45°) = \dfrac{\sqrt{6} + \sqrt{2}}{4}$

5.2 - Sum and Difference Identities

Sine/Cosine Identities

Sum/Difference of Angles

$$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

$$\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$$

$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

Cofunctions

$$\cos\theta = \sin(\tfrac{\pi}{2} - \theta)$$

$$\sin\theta = \cos(\tfrac{\pi}{2} - \theta)$$

5.2 - Sum and Difference Identities

Deriving the Tangent <u>sum</u> of angles

Start with the Tangent Identity

$$\tan(A+B) = \frac{\sin(A+B)}{\cos(A+B)}$$

$$\tan(A+B) = \frac{\sin A \cos B + \cos A \sin B}{\cos A \cos B - \sin A \sin B}$$

$$\tan(A+B) = \frac{\frac{\sin A \cos B}{\cos A \cos B} + \frac{\cos A \sin B}{\cos A \cos B}}{\frac{\cos A \cos B}{\cos A \cos B} - \frac{\sin A \sin B}{\cos A \cos B}}$$

$$\boxed{\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}}$$

5.2 - Sum and Difference Identities

Deriving the Tangent difference of angles

Start with the Tangent Sum Identity

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

Substitute B with $-B$

$$\tan(A+(-B)) = \frac{\tan A + \tan(-B)}{1 - \tan A \tan(-B)}$$

$$\boxed{\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}}$$

5.2 - Sum and Difference Identities

Sine/Cosine Identities

Sum/Difference of Angles

$$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

$$\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$$

$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

Cofunctions

$$\cos\theta = \sin(\tfrac{\pi}{2} - \theta)$$

$$\sin\theta = \cos(\tfrac{\pi}{2} - \theta)$$

Tangent Identities

$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

$$\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

5.2 - Sum and Difference Identities

$$\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta \qquad \sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta$$

$$\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta \qquad \sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta$$

$$\tan(A+B)=\frac{\tan A+\tan B}{1-\tan A\tan B} \qquad \tan(A-B)=\frac{\tan A-\tan B}{1+\tan A\tan B}$$

Practice - Establish the identity

$$a.\ \frac{\cos(\alpha-\beta)}{\sin\alpha\sin\beta}=\cot\alpha\cot\beta+1$$

$$b.\ \tan\left(\theta+\frac{\pi}{2}\right)=-\cot\theta$$

Ch 5.3 Double-Angle

5.3 - Double-Angle Identities

Deriving the Double Angle Identities

$\sin(2A)$ and $\cos(2A)$

Start with the Sine sum identity

$$\sin(A+B) = \sin A \cos B + \cos A \sin B$$

$$\sin(A+A) = \sin A \cos A + \cos A \sin A$$

$$\boxed{\sin(2A) = 2\sin A \cos A}$$

Start with the Cosine sum identity

$$\cos(A+B) = \cos A \cos B - \sin A \sin B$$

$$\cos(A+A) = \cos A \cos A - \sin A \sin A$$

$$\boxed{\cos(2A) = \cos^2 A - \sin^2 A}$$

5.3 - Double-Angle Identities

Deriving more Double Angle Identities

Start with the Cosine double angle identity

$$\cos(2A) = \cos^2 A - \sin^2 A$$

$$\cos(2A) = (1 - \sin^2 A) - \sin^2 A$$

$$\boxed{\cos(2A) = 1 - 2\sin^2 A}$$

Start with the Cosine double angle identity

$$\cos(2A) = \cos^2 A - \sin^2 A$$

$$\cos(2A) = \cos^2 A - (1 - \cos^2 A)$$

$$\boxed{\cos(2A) = 2\cos^2 A - 1}$$

5.3 - Double-Angle Identities

Deriving the Tangent Double Angle Identity

$$\tan(2A) = \frac{\sin(2A)}{\cos(2A)}$$

Or start with the Tangent sum identity

$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

$$\tan(A + A) = \frac{\tan A + \tan A}{1 - \tan A \tan A}$$

$$\tan(2A) = \frac{2 \tan A}{1 - \tan^2 A}$$

5.3 - Double-Angle Identities

If $\sin x = -\frac{4}{5}$ and $\cos x < 0$, find $\sin(2x)$, $\cos(2x)$, and $\tan(2x)$.

$$\sin^2 x + \cos^2 x = 1$$

$$\left(-\frac{4}{5}\right)^2 + \cos^2 x = 1$$

$$\cos^2 x = \frac{9}{25}$$

$$\cos x = -\sqrt{\frac{9}{25}} = -\frac{3}{5}$$

$$\sin(2x) = 2\sin x \cos x$$

$$\sin(2x) = 2\left(-\frac{4}{5}\right)\left(-\frac{3}{5}\right)$$

$$\sin(2x) = \frac{24}{25}$$

$$\cos(2x) = \cos^2 x - \sin^2 x$$

$$\cos(2x) = \left(-\frac{3}{5}\right)^2 - \left(-\frac{4}{5}\right)^2$$

$$\cos(2x) = -\frac{7}{25}$$

$$\tan(2x) = \frac{\sin(2x)}{\cos(2x)}$$

$$\tan(2x) = \frac{24/25}{-7/25}$$

$$\tan(2x) = -\frac{24}{7}$$

5.3 - Double-Angle Identities

$$\sin(2A) = 2\sin A\cos A$$

$$\cos(2A) = \cos^2 A - \sin^2 A$$

$$\tan(2A) = \frac{2\tan A}{1 - \tan^2 A}$$

Practice

Simplify

1. $\dfrac{1 - \cos 2x}{1 + \cos 2x}$

2. $\dfrac{1 + \sin x - \cos 2x}{\cos x + \sin 2x}$

$\tan^2 x$

$\tan x$

5.3 - Double-Angle Identities

Practice

If $\cos x = \frac{3}{5}$ and $\sin x < 0$, find $\sin(2x)$, $\cos(2x)$, and $\tan(2x)$.

$$\sin(2x) = -\frac{24}{25}$$

$$\cos(2x) = -\frac{7}{25}$$

$$\tan(2x) = \frac{24}{7}$$

$$\sin(2A) = 2\sin A\cos A$$

$$\cos(2A) = \cos^2 A - \sin^2 A$$

$$\tan(2A) = \frac{2\tan A}{1 - \tan^2 A}$$

5.3 - Double-Angle Identities

$\sin(2A) = 2\sin A\cos A$

$\cos(2A) = \cos^2 A - \sin^2 A$

$\cos(2A) = 1 - 2\sin^2 A$

$\cos(2A) = 2\cos^2 A - 1$

Practice - Establish the identity

$a.\ (\sin x - \cos x)^2 = 1 - \sin(2x)$

$b.\ \cos(3x) = (1 - 4\sin^2 x)\cos x$

5.1 - Trigonometric Identities

Practice - Simplify

$$y = \frac{\cot x - \tan x}{\cot x + \tan x}$$

$$y = \cos(2x)$$

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